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Some soluble models for periodic decay and revival

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Abstract. We present three analytically tractable models for a two-level atom interacting with a coherent or thermal radiation field and study the behaviour of the atomic excitation energy for various initial conditions.

1. Introduction

It has recently been demonstrated (Eberly *et al* 1980, Narozhny *et al* 1981) that the dynamical behaviour of a two-level atom coupled to a coherent radiation field shows interesting phenomena of decay and regeneration. The expectation value of the atomic energy operator $\langle H_A \rangle$, in this simple example of an interacting system, was long thought to be an essentially decaying function of time; but Eberly and co-workers (Eberly *et al* 1980) found by numerical methods that $\langle H_A \rangle$ in fact revives many times to nearly its initial value.

Eberly *et al* (1980) used the model of Jaynes and Cummings (1963) in which the time dependence of H_A is determined by a frequency operator Ω proportional to the square root of the number operator N for the radiation quanta. This obscures the periodic features inherent in such systems and we later proposed alternative models (Sukumar and Buck 1981, Buck and Sukumar 1981), in which Ω has a linear dependence on N, so that regeneration could be studied analytically for both coherent and thermal radiation. Here we give a fuller account of our work.

In § 2 we present several loss-free models for a two-level atom coupled to single mode radiation, show that the equations of motion are soluble, and investigate the circumstances under which the expectation values of suitable operators decay and revive. Section 3 contains a derivation of general expressions for the atomic energy expectation value when the initial radiation is either coherent or described by a thermal density matrix. In § 4 the solutions of the equations of motion given in § 2 are used to study the dynamics of atom and field evolving from different initial conditions and § 5 summarises our conclusions.

We adopt essentially the same notation as in the preceding paper (Buck and Sukumar 1984) except that $\boldsymbol{J}$, J_3, $J_\pm$ are replaced by the spin $\frac{1}{2}$ operators $\boldsymbol{S}$, S_3, $S_\pm$ since we are here concerned only with two-level atoms.

2. Models

2.1. Basic relations

We consider an atom with energy levels $\pm(\frac{1}{2}\varepsilon)$ coupled to a single radiation mode of unit frequency. In terms of spin-$\frac{1}{2}$ operators $\boldsymbol{S}$ the atomic Hamiltonian is represented by εS_3 while transitions are described by $S_\pm = S_1 \pm iS_2$. The field energy is taken to be $a^\dagger a = N$, the number operator for oscillator phonons, and the various models we discuss are distinguished by the types of coupling allowed and the number of field quanta mediating an atomic transition.

The non-vanishing equal time commutators are

$$[S_+, S_-] = 2S_3, \qquad [S_3, S_\pm] = \pm S_\pm, \qquad [a, a^\dagger] = 1, \tag{1}$$

and the following algebraic and anti-commutator relations hold at all times

$$\begin{aligned} &\boldsymbol{S}^2 = \tfrac{3}{4}, \qquad S_3^2 = \tfrac{1}{4}, \\ &S_+S_- = \boldsymbol{S}^2 - S_3^2 + S_3 = \tfrac{1}{2} + S_3, \qquad S_-S_+ = \boldsymbol{S}^2 - S_3^2 - S_3 = \tfrac{1}{2} - S_3, \\ &\{S_+, S_-\} = 1, \qquad \{S_\pm, S_3\} = 0. \end{aligned} \tag{2}$$

The model Hamiltonians described below are designed to yield a fundamental frequency operator Ω which is a linear function of N.

2.2. *m-phonon transitions*

We begin with Hamiltonians of the form

$$H = a^\dagger a + (\varepsilon + \mu a^\dagger a)S_3 + (\tfrac{1}{2}\lambda)[S_+(a)^m + S_-(a^\dagger)^m], \tag{3}$$

where ε, μ and λ are constants and m is a positive integer denoting the number of phonons absorbed or emitted as the atom is excited or de-excited. These models differ from those considered by Sukumar and Buck (1981) by the addition of a term $\mu a^\dagger a S_3$ which represents an intensity dependent level shift. The original Jaynes–Cummings (1963) model has $\mu = 0$ and $m = 1$.

Using equations (1) and (2) we can write down the Heisenberg equations $i\dot{Q} = [Q, H]$ for various operators Q. It is useful to define a constant

$$\alpha = \varepsilon - m \tag{4}$$

measuring the amount of detuning from resonance, and the following subsidiary operator combinations:

$$A = H - C + \tfrac{1}{4}\mu m, \qquad B = (\tfrac{1}{2}\lambda)[S_+(a)^m - S_-(a^\dagger)^m], \tag{5a, b}$$

$$C = a^\dagger a + mS_3, \qquad D = (\tfrac{1}{2}\lambda)[S_+(a)^m + S_-(a^\dagger)^m] = A - ES_3, \tag{5c, d}$$

$$E = \alpha + \mu C, \qquad F = (C + \tfrac{1}{2}m)(C + \tfrac{1}{2}m - 1)\ldots(C + 1 - \tfrac{1}{2}m), \tag{5e, f}$$

where F has m factors altogether. The operators C, E and F commute with every term in H and are therefore constants of the motion. Clearly, A is also a constant operator.

Thus on rewriting H in the form

$$H = (C - \tfrac{1}{4}\mu m) + ES_3 + D, \tag{6}$$

and noting the relations

$$[S_3, B] = D, \qquad [S_3, D] = B, \tag{7}$$

it is quickly verified that

$$i\dot{S}_3 = B \qquad \text{and} \qquad i\dot{B} = -ED + [B, D]. \tag{8}$$

In an earlier paper (Buck and Sukumar 1981) it was shown that

$$[B, D]=\lambda^2 F S_3, \tag{9}$$

so from equations (8) and (5*d*) we find the result

$$\ddot{S}_3+\Omega^2 S_3=EA, \tag{10}$$

where the effective frequency operator Ω is given by

$$\Omega^2=E^2+\lambda^2 F. \tag{11}$$

The equation of motion (10) for $S_3(t)$ involves only the constant operators E, A and Ω whose expectation values may be calculated at $t=0$. Its general solution is easily seen to be

$$S_3(t)=S_3(0)\cos\Omega t+(\dot{S}_3(0)/\Omega)\sin\Omega t+(EA/\Omega^2)[1-\cos\Omega t], \tag{12}$$

which, using equations (5), (8) and (11) may also be written as

$$S_3(t)=S_3(0)-(\mathrm{i}B(0)/\Omega)\sin\Omega t+[(ED(0)-\lambda^2 FS_3(0))/\Omega^2][1-\cos\Omega t]. \tag{13}$$

For the above type of Hamiltonian we now write down the special conditions under which Ω depends linearly on C, this leading to an analytic treatment of periodic decay and regeneration in the dynamics of $\langle S_3(t)\rangle$.

Model 1. With

$$m=1, \tag{14}$$

and

$$\lambda^2=2(\mu^2-2\alpha\mu), \tag{15}$$

we have

$$\Omega=\pm[\mu(C+1)-\alpha]. \tag{16}$$

Model 2. Putting

$$m=2 \tag{17}$$

and

$$\lambda^2=4\alpha(\alpha-\mu), \tag{18}$$

gives

$$\Omega=\pm[(2\alpha-\mu)C+\alpha]. \tag{19}$$

2.3. *Intensity-dependent coupling*

A third model, with one-phonon transitions ($m=1$) and Ω linear in C, can be constructed from the modified Hamiltonian

$$\tilde{H}=a^{+}a+(\varepsilon+\mu a^{+}a)S_3+(\tfrac{1}{2}\lambda)[S_{+}a\sqrt{a^{+}a}+S_{-}\sqrt{a^{+}a}\,a^{+}], \tag{20}$$

which differs from the model of Buck and Sukumar (1981) by the presence of the term in μ. Defining $\alpha=\varepsilon-1$, $\tilde{A}=\tilde{H}-C+\frac{1}{4}\mu$, $C=a^{+}a+S_3$ and $E=\alpha+\mu C$, the equation of motion for S_3 is

$$\ddot{S}_3+\Omega^2 S_3=E\tilde{A} \tag{21}$$

where

$$\Omega^2=E^2+\lambda^2(C+\tfrac{1}{2})^2. \tag{22}$$

The general solution of equation (21) has the form

$$S_3(t)=S_3(0)-(\mathrm{i}B(0)/\Omega)\sin\Omega t+[(ED(0)-\lambda^2(C+\tfrac{1}{2})^2S_3(0))/\Omega^2][1-\cos\Omega t], \tag{23}$$

where

$$B=(\tfrac{1}{2}\lambda)[S_+a\sqrt{a^\dagger a}-S_-\sqrt{a^\dagger a}\,a^\dagger], \tag{24}$$

and

$$D=(\tfrac{1}{2}\lambda)[S_+a\sqrt{a^\dagger a}+S_-\sqrt{a^\dagger a}\,a^\dagger]. \tag{25}$$

We now impose special constraints to obtain:

Model 3. With

$$m=1 \tag{26}$$

and

$$\mu=2\alpha, \tag{27}$$

giving

$$\Omega=\pm(\lambda^2+\mu^2)^{1/2}(C+\tfrac{1}{2}). \tag{28}$$

2.4. Basis states

The atomic states are denoted by $|M\rangle$, $M=\pm\frac{1}{2}$, and satisfy $S_3(0)|M\rangle=M|M\rangle$ while the field phonon states $|n\rangle$, $n=0,1,2,\ldots$, are eigenstates of $a^\dagger(0)a(0)=N(0)$. Clearly the states $|n\rangle|M\rangle$ are eigenfunctions of the operator $C_m=a^\dagger a+mS_3$ with eigenvalues

$$\gamma_{n,M}=(n+mM) \tag{29}$$

so that the eigenvalues of $\Omega(C_m)$ are always linear functions $\omega(\gamma_{n,M})$ of $\gamma_{n,M}$.

In later sections we shall need only diagonal matrix elements of S_3 with respect to the states $|n\rangle|M\rangle$ and it is obvious from the definitions of B and D that

$$\langle nM|B(0)|nM\rangle=0, \qquad \langle nM|ED(0)|nM\rangle=0. \tag{30}$$

Hence the above operators may be dropped from equations (13) and (23). Also in these latter equations we shall denote the coefficients of $S_3(0)$ in the factors multiplying $[1-\cos\Omega t]$ by G_i ($i=1$, 2, 3 for the three models). They are given explicitly by

$$G_1=\lambda^2(C_1+\tfrac{1}{2})/\Omega_1^2, \qquad \Omega_1=\pm[\mu C_1+(\mu-\alpha)], \tag{31}$$

$$G_2=\lambda^2C_2(C_2+1)/\Omega_2^2, \qquad \Omega_2=\pm[(2\alpha-\mu)C_2+\alpha], \tag{32}$$

$$G_3=\lambda^2/(\lambda^2+\mu^2), \qquad \Omega_3=\pm(\lambda^2+\mu^2)^{1/2}(C_1+\tfrac{1}{2}), \tag{33}$$

with eigenvalues represented by $g_i(\gamma_{n,M})$ or, equivalently, by $g_i(\omega_{n,M})$.

3. Initial conditions

3.1. Thermal radiation

Radiation in thermal equilibrium at temperature $T_R = (k\beta_R)^{-1}$ is described by the density matrix

$$\rho_R = [1 - \exp(-\beta_R)] \sum_{n=0}^{\infty} |n\rangle \exp(-n\beta_R)\langle n|. \tag{34}$$

If at $t = 0$ the atom is in a definite state $|M\rangle$ and the phonons are thermalised then for $t > 0$ the expectation value of $S_3(t)$ is

$$\langle S_3(t)\rangle_{\beta_R, M} = [1 - \exp(-\beta_R)] \sum_{n=0}^{\infty} \exp(-n\beta_R)\langle nM|S_3(t)|nM\rangle, \tag{35}$$

where the matrix element has the form

$$\langle nM|S_3(t)|nM\rangle = M[1 + g_i(\omega_{n,M})\{\cos(\omega_{n,M}t) - 1\}], \tag{36}$$

as follows from equations (13), (23) and (30) ... (33). The average energy of the atom is of course $\varepsilon\langle S_3(t)\rangle_{\beta_R, M}$.

Here and in later sections we add suffixes to $\langle S_3(t)\rangle$ to indicate the initial conditions of field and atom.

When the atom is also initially in thermal equilibrium, at temperature $T_A = (k\beta_A)^{-1}$, its density matrix is given by

$$\rho_A = \sum_M |M\rangle \frac{\exp(-\varepsilon M\beta_A)}{2\cosh(\frac{1}{2}\varepsilon\beta_A)}\langle M|, \tag{37}$$

and the joint density matrix of the system is

$$\rho = \rho_R \rho_A. \tag{38}$$

The expression for $\langle S_3(t)\rangle$ when $t > 0$ is then

$$\langle S(t)\rangle_{\beta_R, \beta_A} = \frac{\Sigma_M \exp(-\varepsilon M\beta_A)\langle S_3(t)\rangle_{\beta_R, M}}{2\cosh(\frac{1}{2}\varepsilon\beta_A)} \tag{39}$$

3.2. Coherent radiation

A coherent state $|Z\rangle$ of the radiation field is defined as an eigenstate of the destruction operator a. It has the normalised form

$$|Z\rangle = \exp(-\tfrac{1}{2}|Z|^2) \sum_{n=0}^{\infty} \frac{Z^n}{\sqrt{n!}}|n\rangle, \tag{40}$$

where the eigenvalue Z is an arbitrary complex number. The mean number of quanta in the state $|Z\rangle$ is given by

$$\nu = \langle Z|a^\dagger a|Z\rangle = |Z|^2. \tag{41}$$

If at $t = 0$ the atom is in state $|M\rangle$ and the radiation is coherent with average phonon number ν then at later times the expectation value of $S_3(t)$ is expressed by

$$\langle S_3(t)\rangle_{\nu, M} = M\,e^{-\nu} \sum_{n=0}^{\infty} \frac{\nu^n}{n!}[1 + g_i(\omega_{n,M})\{\cos(\omega_{n,M}t) - 1\}]. \tag{42}$$

4. Analysis of $\langle S_3(t)\rangle$

The eigenvalues $\gamma_{n,M}$ of the operator $C_m = a^\dagger a + mS_3$ given by $\gamma_{n,M} = n + Mm$ satisfy the relations

$$\gamma_{n,-1/2} = \gamma_{n-m,1/2} = n - \tfrac{1}{2}m. \tag{43}$$

Hence the eigenvalue of Ω satisfies

$$\omega_{n,-1/2} = \omega_{n-m,1/2} \tag{44}$$

which in turn implies that

$$g(\omega_{n,-1/2}) = g(\omega_{n-m,1/2}). \tag{45}$$

The index m depends upon the specific model. Here we keep the discussion general. The above equalities and equations (35), (36), (39) and (42) lead to the following expressions for $\langle S_3(t)\rangle$

$$\langle S_3(t)\rangle_{\beta_{\mathrm{R}},-1/2} = -\tfrac{1}{2}[h_1(t)+1], \tag{46a}$$

$$\langle S_3(t)\rangle_{\beta_{\mathrm{R}},+1/2} = +\tfrac{1}{2}[\mathrm{e}^{m\beta_{\mathrm{R}}}h_1(t)+1], \tag{46b}$$

$$\langle S_3(t)\rangle_{\beta_{\mathrm{R}},\beta_{\mathrm{A}}} = -\tfrac{1}{2}\tanh(\tfrac{1}{2}\beta_{\mathrm{A}}\varepsilon) + \tfrac{1}{2}\exp(m\beta_{\mathrm{R}}/2)\frac{\sinh[\frac{1}{2}(m\beta_{\mathrm{R}}-\beta_{\mathrm{A}}\varepsilon)]}{\cosh(\frac{1}{2}\beta_{\mathrm{A}}\varepsilon)}h_1(t), \tag{46c}$$

$$\langle S_3(t)\rangle_{\nu,M} = M[1+h_2(t)] \tag{46d}$$

where

$$h_1(t) = [1-\exp(-\beta_{\mathrm{R}})]\sum_{n=0}^{\infty}\mathrm{e}^{-n\beta_{\mathrm{R}}}g_i(\omega_{n,-1/2})[\cos(\omega_{n,-1/2}t)-1] \tag{47}$$

and

$$h_2(t) = \mathrm{e}^{-\nu}\sum_{n=0}^{\infty}\frac{\nu^n}{n!}g_i(\omega_{n,-1/2})[\cos(\omega_{n,-1/2}t)-1]. \tag{48}$$

By inspection we find that

$$h_1(0)=0, \qquad h_1(t)\leqslant 0, \qquad \langle S_3(0)\rangle_{\beta_{\mathrm{R}}\beta_{\mathrm{A}}} = -\tfrac{1}{2}\tanh(\tfrac{1}{2}\beta_{\mathrm{A}}\varepsilon).$$

The restrictions implied by the above equations lead to the following observations.

(i) If $\beta_{\mathrm{A}}\varepsilon > m\beta_{\mathrm{R}}$ then $\langle S_3(0)\rangle$ is the minimum value of $\langle S_3(t)\rangle$.

(ii) If $\beta_{\mathrm{A}}\varepsilon < m\beta_{\mathrm{R}}$ then $\langle S_3(0)\rangle$ is the maximum value of $\langle S_3(t)\rangle$.

(iii) If $\beta_{\mathrm{A}}\varepsilon = m\beta_{\mathrm{R}}$ then $\langle S_3(t)\rangle = \langle S_3(0)\rangle$ which shows that $\langle S_3(t)\rangle$ does not change with time. There are two distinct ways in which the conditions $\beta_{\mathrm{A}}\varepsilon = m\beta_{\mathrm{R}}$ can be met. If $\varepsilon = m$ and $\beta_{\mathrm{A}} = \beta_{\mathrm{R}}$, the atom and the field are at the same temperature at $t=0$ and remain in thermal equilibrium at later times. If $\beta_{\mathrm{A}} \neq \beta_{\mathrm{R}}$ and $\varepsilon \neq m$ but $\beta_{\mathrm{A}}\varepsilon = m\beta_{\mathrm{R}}$, $\langle S_3(t)\rangle$ is still a constant. This second possibility implies that even though the atom and the field are not at the same temperature at $t=0$, $\langle S_3(t)\rangle$ is a constant in spite of the interaction.

We now consider in turn each of the three models constructed in § 2, and examine the special features that arise in the dynamics of $\langle S_3(t)\rangle$.

Model 1. In this model the atomic transition is mediated by a single phonon ($m=1$) and the following relations apply.

$$C_m = C_1 = a^\dagger a + S_3, \qquad \gamma_{n,-1/2} = \gamma_{n-1,1/2} = n - \tfrac{1}{2}, \tag{49a, b}$$

$$\omega_{n,-1/2} = \omega_{n-1,1/2} = \mu(n+\xi), \tag{49c}$$

$$g(\omega_{n,-1/2}) = g(\omega_{n-1,+1/2}) = 4\xi n/(n+\xi)^2 \tag{49d}$$

in which

$$\xi = -\tfrac{1}{2} - \alpha/\mu = \lambda^2/4\mu^2. \tag{50}$$

It is convenient to define the scaled time $\tau = \mu t$ so that $\omega_{n-1/2}t = (n+\xi)\tau$. In terms of these new variables we get from equations (47) and (48)

$$h_1(\tau) = (1 - e^{-\beta_R}) \sum_{n=0}^{\infty} e^{-n\beta_R} \frac{4\xi n}{(n+\xi)^2}[\cos(n+\xi)\tau - 1], \tag{51}$$

$$h_2(\tau) = e^{-\nu} \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \frac{4\xi n}{(n+\xi)^2}[\cos(n+\xi)\tau - 1]. \tag{52}$$

Inspection of $h_1(\tau)$ and $h_2(\tau)$ enables the identification of the periodic features of this model. If ξ is a rational number of the form $\xi = J/K$ then each term in the series representations of h_1 and h_2 is periodic in τ with period $2\pi K$. Since h_1 and h_2 are periodic functions of τ exact revivals of $\langle S_3(\tau)\rangle$ occur with period $2\pi K$.

$h_1(\tau)$ can also be written in the form

$$h_1(\tau) = 4\xi(1 - e^{-\beta_R}) \frac{\partial}{\partial \beta_R} \int_0^\tau dt' \int_0^{t'} dt''\, P(t'') \tag{53}$$

in which

$$P(\tau) = \sum_{n=0}^{\infty} e^{-n\beta_R} \cos(n+\xi)\tau = \frac{\cos \xi\tau - e^{-\beta_R}\cos(\xi-1)\tau}{1 + e^{2\beta_R} - 2\,e^{\beta_R}\cos\tau}. \tag{54}$$

The series for $h_2(\tau)$ is not exactly summable. However, when the mean phonon number in the field, ν is large, the series can be summed approximately using the saddlepoint approximation which is accurate to order $1/\nu$. The resulting estimate is

$$h_2(\tau) = -(4\xi/\nu)[1 - \exp(-2\nu \sin^2 \tfrac{1}{2}\tau)\cos(\nu \sin\tau + \tau\xi - \tau)]. \tag{55}$$

This expression clearly shows that h_2 collapses rapidly if ν is large, but revives eventually. The revivals are exact if ξ is a rational number. Even when ξ is not a rational number, partial revivals will occur because of revivals of the envelope function $\exp(-2\nu \sin^2 \tfrac{1}{2}\tau)$.

Model 2. In this model, the interaction of the atom with the radiation field is mediated by two oscillator quanta ($m=2$) and the following relations apply.

$$C_m = C_2 = a^\dagger a + 2S_3, \qquad \gamma_{n,-1/2} = \gamma_{n-2,1/2} = n - 1, \tag{56a, b}$$

$$\omega_{n,-1/2} = \omega_{n-2,1/2} = [(4\alpha^2 + \lambda^2)/4\alpha](n - \xi),$$

$$g(\omega_{n,-1/2}) = g(\omega_{n-2,1/2}) = [\lambda^2\alpha^2/(\alpha^2 + \tfrac{1}{4}\lambda^2)^2][n(n-1)/(n-\xi)^2] \tag{56c, d}$$

in which

$$\xi = \lambda^2/(4\alpha^2+\lambda^2). \tag{57}$$

In terms of a scaled time defined by $\tau=[(4\alpha^2+\lambda^2)/4\alpha]t$, so that $\omega_{n,-1/2}t=(n-\xi)\tau$, we get from equations (47) and (48)

$$h_1(\tau)=\frac{\lambda^2\alpha^2}{(\frac{1}{4}\lambda^2+\alpha^2)^2}(1-e^{-\beta_R})\sum_{n=0}^{\infty} e^{-n\beta_R}\frac{n(n-1)}{(n-\xi)^2}[\cos(n-\xi)\tau-1], \tag{58}$$

$$h_2(\tau)=\frac{\lambda^2\alpha^2}{(\frac{1}{4}\lambda^2+\alpha^2)^2}e^{-\nu}\sum_{n=0}^{\infty}\frac{\nu^n}{n!}\frac{n(n-1)}{(n-\xi)^2}[\cos(n-\xi)\tau-1]. \tag{59}$$

Inspection of $h_1(\tau)$ and $h_2(\tau)$ shows that if ξ is a rational number of the form J/K then h_1 and h_2 are periodic with period $2\pi K$ since each term in the series is a periodic function of τ. Thus exact revivals and decay of $\langle S_3(\tau)\rangle$ will occur with period $2\pi K$.

$h_1(\tau)$ can also be written in the convenient form

$$h_1(\tau)=\frac{\lambda^2\alpha^2}{(\frac{1}{4}\lambda^2+\alpha^2)^2}[1-\exp(-\beta_R)]$$

$$\times\left[P(\tau)+(2\xi-1)\int_0^\tau Q(t')\,dt'+\xi(1-\xi)\int_0^\tau dt\int_0^t dt'\,P(t')\right] \tag{60}$$

in which

$$P(\tau)=\sum_{n=0}^{\infty}\exp(-n\beta_R)\cos(n-\xi)\tau=\frac{\cos\xi\tau-\exp(-\beta_R)\cos(\xi+1)\tau}{1+\exp(-2\beta_R)-2\exp(-\beta_R)\cos\tau}, \tag{61}$$

$$Q(\tau)=-\tau^{-1}\,\partial P(\xi,\tau)/\partial\xi. \tag{62}$$

h_2 can be written in the form

$$h_2(\tau)=\frac{\lambda^2\alpha^2}{(\frac{1}{4}\lambda^2+\alpha^2)^2}e^{-\nu}\sum_{n=0}^{\infty}\frac{\nu^n}{n!}[\cos(n-\xi)\tau-1]$$

$$+\frac{4\lambda^2\alpha^2}{(\lambda^2+\alpha^2)^2}e^{-\nu}\sum_{n=0}^{\infty}\frac{\nu^n}{n!}\frac{(2\xi-1)n-\xi^2}{(n-\xi)^2}[\cos(n-\xi)\tau-1]. \tag{63}$$

Summing of the first series exactly and estimation of the second series by saddlepoint approximation leads to the expression

$$h_2(\tau)=[\lambda^2\alpha^2/(\tfrac{1}{4}\lambda^2+\alpha^2)^2]\{\exp(-2\nu\sin^2\tfrac{1}{2}\tau)\cos(\nu\sin\tau-\xi\tau)-1$$
$$+[(2\xi-1)/\beta][\exp(-2\nu\sin^2\tfrac{1}{2}\tau)\cos(\nu\sin\tau-\xi\tau-\tau)-1]\}$$
$$+O(1/\nu^2). \tag{64}$$

The periodic character of $\exp(-23\nu\sin^2\frac{1}{2}\tau)$ implies that even though $h_2(\tau)$ collapses rapidly for large ν and small τ revivals of $h_2(\tau)$ occur for all values of ξ. The revivals can be total or partial depending upon whether ξ is a rational number or not.

For the special value $\xi=\frac{1}{2}$ corresponding to the choice of parameters $\mu=0$, $\lambda=\pm2\alpha$ referred to in an earlier paper (Sukumar and Buck 1981), the following expressions result

$$h_1(\tau) = (1-\exp(-\beta_R))\left[\frac{\exp(-\beta_R)(1-\exp(-\beta_R))\cos\frac{1}{2}\tau}{1+\exp(-2\beta_R)-2\exp(-\beta_R)\cos\tau} - \frac{\exp(-\beta_R)}{1-\exp(-\beta_R)}\right.$$

$$\left. +\tfrac{1}{4}\exp(-\tfrac{1}{2}\beta_R)\int_0^\tau \mathrm{d}t' \tan^{-1}\frac{\sin\frac{1}{2}t'}{\sinh(\frac{1}{2}\beta_R)}\right], \tag{65}$$

$$h_2(\tau) = [\exp(-2\nu\sin^2\tfrac{1}{2}\tau)\cos(\nu\sin\tau - \tfrac{1}{2}\tau) - 1]$$
$$+ (1/4\nu^2)[\exp(-2\nu\sin^2\tfrac{1}{2}\tau)\cos(\nu\sin\tau - \tfrac{5}{2}\tau) - 1] + O(1/\nu^3). \tag{66}$$

Model 3. In this model, single oscillator quanta ($m=1$) mediate the atom–field interaction leading to the following equations.

$$C_m = C_1 = a^\dagger a + S_3, \qquad \gamma_{n,-1/2} = \gamma_{n-1,1/2} = n - 1/2, \tag{67a, b}$$

$$\omega_{n,-1/2} = \omega_{n-1,1/2} = (\lambda^2 + \mu^2)^{1/2} n,$$

$$g(\omega_{n,-1/2}) = g(\omega_{n-1,1/2}) = \lambda^2/(\lambda^2+\mu^2). \tag{67c, d}$$

In terms of the scaled time τ defined by $\tau = (\lambda^2+\mu^2)^{1/2}t$, so that $\omega_{n,-1/2}t = n\tau$, we get from equations (47) and (48)

$$h_1(\tau) = \frac{\lambda^2}{\lambda^2+\mu^2}[1-\exp(-\beta_R)]\sum_{n=0}^{\infty}\exp(-n\beta_R)(\cos n\tau - 1), \tag{68}$$

$$h_2(\tau) = \frac{\lambda^2}{\lambda^2+\mu^2}\mathrm{e}^{-\nu}\sum_{n=0}^{\infty}\frac{\nu^n}{n!}(\cos n\tau - 1). \tag{69}$$

The series h_1 and h_2 can be summed exactly to give

$$h_1(\tau) = \frac{\lambda^2}{\lambda^2+\mu^2}\frac{[\exp(\beta_R)+1](\cos\tau - 1)}{\exp(2\beta_R) - 2\exp(\beta_R)\cos\tau + 1}, \tag{70}$$

$$h_2(\tau) = [\lambda^2/(\lambda^2+\mu^2)][\exp(-2\nu\sin^2\tfrac{1}{2}\tau)\cos(\nu\sin(\tau)+\tau) - 1]. \tag{71}$$

These expressions clearly show that h_1 and h_2 are periodic in τ with period 2π leading to exact revivals and decay in the dynamics of $S_3(t)$.

5. Conclusions

For thermal initial conditions the following results are common to the three models.

(i) For special values of ε, the resonance tuning parameter, $\langle S_3(t)\rangle$ can remain constant in time even though the atom and the radiation field may not be at the same temperature at $t=0$.

(ii) $\langle S_3(0)\rangle_{\beta_R\beta_A}$ is an extremum value. $\langle S_3(t)\rangle$ does not oscillate about $\langle S_3(0)\rangle$ and can reach the value $\langle S_3(0)\rangle$ only at exact revivals of the system.

(iii) The atom and the radiation field never come to equilibrium through the interaction term in the Hamiltonian if at $t=0$ they are not already in thermal equilibrium.

The model Hamiltonians were so constructed that the expectation values in these models, for different initial conditions, show periodic revival and decay. Our examples clearly show that revivals and decay occur for both coherent state and thermal initial conditions of the field.

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